



TITLE:

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## Keller-Segel 系の爆発解の挙動について

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### 0 Introduction

We consider the behavior of blow-up solutions for (KS)

$$(KS) \quad \begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - \chi u \nabla v) & \text{in } \Omega, t > 0 \\ \tau \frac{\partial v}{\partial t} = \Delta v - \gamma v + \alpha u & \text{in } \Omega, t > 0 \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega, t > 0 \\ u(\cdot, 0) = u_0, v(\cdot, 0) = v_0 & \text{on } \Omega \end{cases}$$

Here  $\Omega$  is a bounded domain in  $\mathbf{R}^2$  with smooth boundary  $\partial\Omega$ , and  $\tau, \chi, \gamma$ , and  $\alpha$  are positive constants, and  $u_0, v_0$  are nonnegative, nontrivial, smooth functions on  $\bar{\Omega}$ .

In what follows we denote  $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p$ ,  $M = \|u_0\|_1$ ,  $\int_{\Omega} f dx = \frac{1}{|\Omega|} \int_{\Omega} f dx$ ,  $D := \{x \in \mathbf{R}^2 \mid |x| < 1\}$ , and let  $T$  be the maximal existence time of solution  $(u, v)$ .

Theorem 1.1 in [1] states:

If  $M < \frac{4\pi}{\alpha\chi}$ , then the solution  $(u, v)$  exists globally in time and globally bounded.

If  $\Omega = \{x \in \mathbf{R}^2 \mid |x| < L\}$  and  $(u_0, v_0)$  is radial in  $x$ , and  $M < \frac{8\pi}{\alpha\chi}$ , then the solution  $(u, v)$  exists globally in time and globally bounded.

Then what happens if  $\frac{4\pi}{\alpha\chi} \leq M < \frac{8\pi}{\alpha\chi}$  and  $(u_0, v_0)$  is non radially symmetric? For simplicity, we put  $\alpha = \gamma = \chi = 1$ , and  $\Omega = D$ .

Theorem 2 in [7] and Lemma 9 in [7] states:

Let  $\tau = 0$ ,  $\Omega = D$ , and  $M < 8\pi$ . If  $T < \infty$ , then there exists  $x_0 \in \partial D$

satisfying

$$\liminf_{t \rightarrow T} \int_{D \cap B(x_0, \epsilon)} u(x, t) dx \geq 4\pi \text{ for any } \epsilon > 0.$$

In this paper, we consider to extend this result to  $\tau > 0$ . A main result is following.

**Theorem** Let  $\tau > 0$ ,  $\Omega = D$ , and  $M < 8\pi$ . If  $T < \infty$ , then there exists a continuous map  $p(t) : [0, T) \rightarrow \partial D$  satisfying

$$\limsup_{t \rightarrow T} \int_{D \cap B(p(t), \epsilon)} u(x, t) dx \geq 2\pi \text{ for any } \epsilon > 0.$$

## 1 Fundamental Lemmas for Theorem

Following Lemmas are known.

**Lemma1** The following holds:

$$\|u(\cdot, t)\|_1 = \|u_0\|_1,$$

and

$$\|v(\cdot, t)\|_1 = e^{-\frac{t}{\tau}} \|v_0\|_1 + \|u_0\|_1 (1 - e^{-\frac{t}{\tau}}).$$

**Lemma2** Put

$$W(t) = \int_{\Omega} u \log u - uv + \frac{1}{2}(|\nabla v|^2 + v^2) dx.$$

Then we have

$$\frac{dW}{dt}(t) + \tau \int_{\Omega} v_i^2 dx + \int_{\Omega} u |\nabla(\log u - v)|^2 dx = 0,$$

and it follows that

$$\frac{dW}{dt}(t) \leq 0, \text{ and } W(t) \leq W(0).$$

**Lemma3** Let  $M = \|u_0\|_1$ . The following holds:

$$a \int_{\Omega} uv dx \leq \int_{\Omega} u \log u dx + M \log \frac{1}{M} \int_{\Omega} e^{av} dx \text{ for any } a > 0.$$

**Lemma4 (Corollary of Proposition[3]-2.3)**

Let  $\Omega = D$ . There exists  $C_{\Omega}$  such that

$$\int_{\Omega} e^w dx \leq C_{\Omega} \exp\left(\frac{1}{8\pi} \|\nabla w\|_2^2 + \frac{1}{|\Omega|} \|w\|_1\right) \text{ for any } 0 \leq w \in W^{1,2}(\Omega).$$

**Proposition[4]-8.1** Let  $F$  be a set of  $w(\cdot, t)$  ( $0 \leq t < T$ ) such that  $t \mapsto w(\cdot, t) \in H^1(D)$  is continuous and  $\sup_{0 \leq t < T} \|w(\cdot, t)\|_{L^1(D)} < \infty$ , then either one of the following holds:

(1) There exists  $\{t_k\} \nearrow T$  such that  $w_k = w(\cdot, t_k) \in F$  satisfying the following.

For any  $\epsilon$ , there exists  $C_\epsilon$  such that

$$\log \left( \int_D e^{w_k} dx \right) \leq \frac{1+\epsilon}{16\pi} \int_D |\nabla w_k|^2 dx + C_\epsilon.$$

(2) There exists a continuous map  $t \mapsto q(t) \in \partial D$  such that

$$\liminf_{t \rightarrow T} \frac{\int_{D \cap B(q(t), \epsilon)} \exp(w(x, t)) P_*(x) dx}{\int_D \exp(w(x, t)) P_*(x) dx} \geq \frac{1}{2} \text{ for any } \epsilon > 0,$$

$$\text{where } P_*(x) = \frac{8}{(1 + |x|^2)^2}.$$

### Brézis-Merle Type Inequality for Parabolic Equations of Second Order

We consider the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + \sum_{j=1}^2 b_j(x, t) \frac{\partial u}{\partial x_j} + c(x, t)u = f & \text{in } \Omega \times (0, T) \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x) & \text{in } \Omega \end{cases}$$

Let  $b_j, c \in H^{\alpha, \frac{\alpha}{2}}(\bar{\Omega} \times [0, T])$  and  $q \in \partial\Omega$ , where  $\alpha$  is a real number with  $0 < \alpha < 1$  and  $h$  belongs to  $H^{\alpha, \frac{\alpha}{2}}(\bar{\Omega} \times [0, T])$  if

$$|h(x, t) - h(y, s)| \leq \text{Const.} (|x - y|^\alpha + |t - s|^{\frac{\alpha}{2}})$$

for any  $(x, t), (y, s) \in \bar{\Omega} \times [0, T]$ . Given  $0 < \tau < T$  and  $0 < \epsilon < 2\pi\nu$ , there exist positive constants  $\eta_0$  with  $\eta_0 \in (0, \frac{1}{4})$  and  $C > 0$  depending on  $\tau, \epsilon, \eta \in (0, \eta_0)$ ,  $\|u_0\|_{L^1(\Omega)}$ , and  $\|f\|_{L^1(\Omega \times (0, T))}$  such that  $\eta \in (0, \eta_0)$  and  $\sup_{0 < t < T} \|f^+(t)\|_{L^1(\Omega \cap B(q, 3\eta))} \leq 2\pi\nu - \epsilon$  imply

$$\int_{B(q, \eta)} e^{u(x, t)} dx \leq C \text{ for } \tau \leq t \leq T,$$

where  $u$  denote the solution of the above problem.

**Proposition1** The following holds:

(1)  $T < \infty$  implies  $\lim_{t \rightarrow T} \int_{\Omega} u v dx = \infty$ .

- (2)  $T < \infty$  implies  $\lim_{t \rightarrow T} \int_{\Omega} e^{av} dx = \infty$  for any  $a > \frac{M + \sqrt{M^2 - 4\pi M}}{2M}$ .
- (3)  $T < \infty$  implies  $\lim_{t \rightarrow T} \int_{\Omega} |\nabla v|^2 dx = \infty$ .

## 2 Proof of Proposition1

Before proving Proposition1, we remark that  $T < \infty$  implies  $M \geq 4\pi$  by the contraposition of Theorem1.1 in [1], so in the root sign  $M^2 - 4\pi M$  is not negative.

### Proof of Proposition1

Theorem1 in [5] shows that  $T < \infty$  implies

$$\lim_{t \rightarrow T} \|uv\|_1 = \lim_{t \rightarrow T} \|e^{av}\|_1 = \lim_{t \rightarrow T} \|\nabla v\|_2^2 = \lim_{t \rightarrow T} \|u \log u\|_1 = \infty \text{ for any } a > 1.$$

So we prove only (2). From Lemma3 and Lemma4 with  $w = av$ , we have

$$a \int_{\Omega} uv dx \leq \int_{\Omega} u \log u dx + \frac{Ma^2}{8\pi} \int_{\Omega} |\nabla v|^2 dx + C \text{ for any } a > 0. \quad (2.1)$$

From Lemma2,

$$\int_{\Omega} u \log u - uv + \frac{1}{2}(|\nabla v|^2 + v^2) dx \leq W(0). \quad (2.2)$$

By (2.1) +  $\frac{Ma^2}{4\pi}(2.2)$ ,

$$\left(a - \frac{Ma^2}{4\pi}\right) \int_{\Omega} uv dx \leq \left(1 - \frac{Ma^2}{4\pi}\right) \int_{\Omega} u \log u dx + C \text{ for any } a > 0.$$

Put  $a = \frac{M + \sqrt{M^2 - 4\pi M}}{M}$  in the above inequality, then

$$\int_{\Omega} u \log u dx \leq \frac{M + \sqrt{M^2 - 4\pi M}}{2M} \int_{\Omega} uv dx + C.$$

Using this and Lemma3, we have

$$\left(a - \frac{M + \sqrt{M^2 - 4\pi M}}{2M}\right) \int_{\Omega} uv dx \leq M \log \frac{1}{M} \int_{\Omega} e^{av} dx + C \text{ for any } a > 0.$$

Since  $\lim_{t \rightarrow T} \int_{\Omega} uv dx = \infty$ ,

$$\lim_{t \rightarrow T} \int_{\Omega} e^{av} dx = \infty \text{ for any } a > \frac{M + \sqrt{M^2 - 4\pi M}}{2M}.$$

**Remark**

1. Proposition 3.1 in [6] shows that  $\|v(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C$  for any  $q \in (1, 2)$ . By using this and Hölder's inequality and Sobolev's imbedding theorem, we have

$$\int_{\Omega} uv dx \leq \|u\|_p \|v\|_{p'} \leq C \|u\|_p \quad \text{for any } p > 1.$$

So, it follows from Proposition 1(1) that  $T < \infty$  implies

$$\lim_{t \rightarrow T} \|u(\cdot, t)\|_p = \infty \quad \text{for any } p > 1.$$

**3 Proof of Theorem****Proof of Theorem**

Suppose the first alternative (1) of Proposition [4]-8.1 holds, then there exists  $\{t_k\} \nearrow T$  such that  $v_k = v(\cdot, t_k)$  satisfy the following:

$$\log\left(\frac{1}{\pi} \int_D e^{v(x, t_k)} dx\right) \leq \frac{1+\epsilon}{16\pi} \int_D |\nabla v(x, t_k)|^2 dx + C_\epsilon \quad \text{for any } \epsilon > 0. \quad (3.1)$$

From Lemma 2 and Lemma 3 with  $a = 1$ , we have

$$\frac{1}{2} \int_D |\nabla v|^2 dx \leq W(0) + M \log \frac{1}{M} \int_D e^v dx \quad (3.2)$$

By  $M(3.1) + (3.2)$ ,

$$\left(\frac{1}{2} - \frac{1+\epsilon}{16\pi} M\right) \int_D |\nabla v(x, t_k)|^2 dx \leq W(0) - M \log M + M \log \pi + C_\epsilon M.$$

Since  $M < 8\pi$ , We can take  $\epsilon$  such that

$$\frac{1}{2} - \frac{1+\epsilon}{16\pi} M > 0.$$

Then

$$\int_D |\nabla v(x, t_k)|^2 dx < \infty.$$

This contradicts to Proposition 1.

Therefore the second alternative (2) of Proposition [4]-8.1 holds. Then there exists a continuous map  $t \in [0, T) \mapsto q(t) \in \partial D$  such that

$$\liminf_{t \rightarrow T} \frac{\int_{D \cap B(q(t), \epsilon)} e^v P_*(x) dx}{\int_D e^v P_*(x) dx} \geq \frac{1}{2} \quad \text{for any } \epsilon > 0. \quad (3.3)$$

Since  $P_*(x) = \frac{8}{(1+|x|^2)^2}$ ,  $x \in D$  implies  $2 \leq P_*(x) \leq 8$ .

From Proposition 1

$$\lim_{t \rightarrow T} \int_D e^v dx = \infty,$$

it follows from (3.3) that

$$\lim_{t \rightarrow T} \int_{D \cap B(q(t), \epsilon)} e^v dx = \infty \text{ for any } \epsilon > 0. \quad (3.4)$$

(a) In case that there exists  $q \in \partial D$  such that  $q(t) \rightarrow q$  ( $t \rightarrow T$ ).

We suppose for this  $q(t)$  there exists  $\eta_1$  such that

$$\limsup_{t \rightarrow T} \int_{D \cap B(q(t), \eta_1)} u dx < 2\pi.$$

Then there exists  $\epsilon > 0$  such that

$$\limsup_{t \rightarrow T} \int_{D \cap B(q(t), \eta_1)} u dx \leq 2\pi - \epsilon,$$

and there exists  $T_0$  such that  $T_0 < t < T$  implies

$$\int_{D \cap B(q(t), \eta_1)} u dx \leq 2\pi - \frac{\epsilon}{2}.$$

Because of the continuity of  $q(t)$ , for this  $\eta_1$  there exists  $T_1$  such that  $t > T_1$  implies  $|q(t) - q| < \frac{\eta_1}{2}$ . Since  $B(q, \frac{\eta_1}{2}) \subset B(q(t), \eta_1)$ ,  $t > \max\{T_0, T_1\} =: T_2$  implies

$$\int_{D \cap B(q, \frac{\eta_1}{2})} u dx \leq 2\pi - \frac{\epsilon}{2}.$$

That is

$$\int_{D \cap B(q, \frac{\eta_1}{2})} u dx \leq 2\pi - \frac{\epsilon}{2} \text{ for any } \eta \in (0, \eta_1).$$

By using Brézis-Merle's inequality, given  $t_0 \in (T_2, T)$  there exists  $\eta_0 \in (0, \min\{\eta_1, \frac{1}{4}\})$  and  $C = C(t_0, \epsilon, \eta) > 0$  ( $\eta \in (0, \eta_0)$ ) such that  $\eta \in (0, \eta_0)$  implies

$$\int_{D \cap B(q, \frac{\eta_1}{8})} e^v dx \leq C \text{ for any } t \in [t_0, T].$$

This contradicts to (3.4). Therefore

$$\limsup_{t \rightarrow T} \int_{D \cap B(q(t), \eta)} u dx \geq 2\pi \text{ for any } \eta > 0.$$

Put  $p(t) = q(t)$ .

(b) In case that there doesn't exist  $q \in \partial D$  such that  $q(t) \rightarrow q$  ( $t \rightarrow T$ ).

Put

$$A := \{\gamma \in \partial D \mid \text{for any } T_0 < T \text{ there exists } t \in (T_0, T) \text{ such that } q(t) = \gamma\}.$$

For any  $\gamma \in A$ , by the definition of  $A$  and (3.4), we have

$$\limsup_{t \rightarrow T} \int_{D \cap B(\gamma, \epsilon)} e^v dx = \infty \text{ for any } \epsilon > 0. \quad (3.5)$$

We suppose for this  $\gamma$  there exists  $\eta_1$  such that

$$\limsup_{t \rightarrow T} \int_{D \cap B(\gamma, \eta_1)} u dx < 2\pi.$$

Then there exists  $\epsilon > 0$  such that

$$\limsup_{t \rightarrow T} \int_{D \cap B(\gamma, \eta_1)} u dx \leq 2\pi - \epsilon,$$

and there exists  $T_0$  such that  $T_0 < t < T$  implies

$$\int_{D \cap B(\gamma, \eta_1)} u dx \leq 2\pi - \frac{\epsilon}{2}.$$

That is

$$\int_{D \cap B(\gamma, \eta)} u dx \leq 2\pi - \frac{\epsilon}{2} \text{ for any } \eta \in (0, \eta_1).$$

By using Brézis-Merle's inequality, given  $t_0 \in (T_0, T)$  there exists  $\eta_0 \in (0, \min\{\eta_1, \frac{1}{4}\})$  and  $C = C(t_0, \epsilon, \eta) > 0$  ( $\eta \in (0, \eta_0)$ ) such that  $\eta \in (0, \eta_0)$  implies

$$\int_{D \cap B(\gamma, \frac{\eta}{3})} e^v dx \leq C \text{ for any } t \in [t_0, T].$$

This contradicts to (3.5). Therefore

$$\limsup_{t \rightarrow T} \int_{D \cap B(\gamma, \eta)} u dx \geq 2\pi \text{ for any } \eta > 0.$$

Put  $p(t) = \gamma$ .

### Remark

1. We use Proposition 1(2) with  $a = 1$  to prove Theorem. But using  $a > \frac{M + \sqrt{M^2 - 4\pi M}}{2M}$ , we can improve the constant  $2\pi$  to a larger one in Theorem, which is now studying.
2. If  $M = 4\pi$ , then  $W(t)$  is bounded from below by putting  $a = 1$ ,  $M = 4\pi$  in (2.1). So when this, it follows from [6] that  $\limsup$  can be changed to  $\liminf$  in Theorem.
3. Theorem is correct even if  $\Omega$  is a simply connected bounded domain in  $\mathbf{R}^2$  with smooth boundary.

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